MATH 3120 & 5120 COURSE NOTES - CLASS MEETING # 1

Math 3120 & 5120 Introduction to PDEs, Spring 2019 Professor: Jared Speck

Class Meeting # 1: Introduction to PDEs

1. What is a PDE?

We will be studying functions $u = u(x^1, x^2, \cdots, x^n)$ and their partial derivatives. Here, $x^1, x^2, \cdots, x^n$ are standard Cartesian coordinates on $\mathbb{R}^n$. We sometimes use the alternate notation $u(x, y)$, $u(x, y, z)$, etc. We also write e.g. $u(r, \theta, \phi)$ for spherical coordinates on $\mathbb{R}^3$, etc. We sometimes also have a "time" coordinate t, in which case $t, x^1, \cdots, x^n$ denotes standard Cartesian coordinates on $\mathbb{R}^{1+n}$. We also use the alternate notation $x^0 \stackrel{\text{def}}{=} t$.

We use lots of different notation for partial derivatives:

$$\frac{\partial}{\partial x^i} u = u_{x^i} = \partial_i u, \qquad 1 \leq i \leq n, \tag{1.0.1a}$$

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} u = u_{x^i x^j} = \partial_i \partial_j u, \qquad 1 \leq i, j \leq n. \tag{1.0.1b}$$

If $i = j$, then we sometimes abbreviate $\partial_i \partial_j u \stackrel{\text{def}}{=} \partial_i^2 u$. If u is a function of (x, y), then we also write $u_x = \frac{\partial}{\partial x} u$, etc.

Definition 1.0.1. A PDE in a single unknown u is an equation involving u and its partial derivatives. All such equations can be written as

$$F(u, u_{x^1}, \cdots, u_{x^n}, u_{x^1 x^1}, \cdots, u_{x^{i_1} \cdots x^{i_N}}, x^1, x^2, \cdots, x^n) = 0, \qquad i_1, \cdots, i_N \in \{1, 2, \cdots, n\} \tag{1.0.2}$$

for some function F.

N is called the *order* of the PDE; N is the maximum number of derivatives appearing in the equation.

Example 1.0.1. With $u = u(t, x)$,

$$-\partial_t^2 u + (1 + \cos u) \partial_x^3 u = 0 \tag{1.0.3}$$

is a third-order nonlinear PDE.

Example 1.0.2. With $u = u(t, x)$,

$$-\partial_t^2 u + 2\partial_x^2 u + u = t \tag{1.0.4}$$

is a second-order linear PDE.

Under suitable assumptions on the functions, g, h_0, h_L, these conditions also lead to a well-posed problem.

3.3. **Robin boundary conditions.** We can also take some linear combinations of the Dirichlet and Neumann conditions:

$$\begin{cases} \partial_t u - D\partial_x^2 u = 0, & (t,x) \in (0,T) \times (0,L), \\ u(0,x) = g(x), & \text{(Cauchy data)}, \\ -\partial_x u(t,0) + \alpha u(t,0) = h_0(t), \qquad \partial_x u(t,L) + \alpha u(t,L) = h_L(t), & \text{(Robin data)}, \end{cases} \tag{3.3.1}$$

where $\alpha > 0$ is a *positive* constant. Under suitable assumptions on the functions, g, h_0, h_L, these conditions also lead to a well-posed problem.

3.4. **Mixed boundary conditions.** The above three boundary conditions are called *homogeneous* because they are of the same type at each end. It is also possible to prescribe one condition at one endpoint, and a different condition at the other endpoint. These are called *mixed boundary conditions*. These conditions also lead to a well-posed problem.

4. Separation of variables

We now discuss a technique, known as *separation of variables*, that can be used to explicitly solve certain PDEs. It is especially useful in the study of certain linear PDEs, though it is unfortunately far from universally applicable.

In a nutshell, the separation of variables technique can be summarized as:

- Look for a solution of the form $u(t,x) = v(t)w(x)$.
- Plug this guess into the PDE and hope that the PDE forces the functions v and w to be solutions to ODEs that can be solved without too much trouble.

As we will see, when one tries to apply this technique, one quickly runs into difficulties that are best addressed using techniques from Fourier analysis. We don't have time right now to give a detailed introduction to Fourier analysis, but we will return to it later in the course if time permits; at the moment, we will only show how to use some of these techniques, without fully justifying them.

A great way to illustrate separation of variables is through an example. Let's try to solve the heat equation problem with homogeneous (i.e., vanishing) Dirichlet conditions

$$\begin{cases} u_t - u_{xx} = 0, & (t,x) \in (0,T] \times [0,1], \\ u(0,x) = x, & x \in [0,1], \\ u(t,0) = 0, \qquad u(t,1) = 0, \end{cases} \tag{4.0.1}$$

by separation of variables.

Remark 4.0.1. Note that such a solution cannot possibly be continuous at the point $(t,x) = (0,1)$.

We plug in the form $u(t,x) = v(t)w(x)$ into (4.0.1) and discover that

$$\frac{v'(t)}{v(t)} = \frac{w''(x)}{w(x)}. \tag{4.0.2}$$

MATH 3120 # 5120 COURSE NOTES - CLASS MEETING # 5

Math 3120 # 5120 Introduction to PDEs, Spring 2019 Professor: Jared Speck

Class Meeting # 5: The Fundamental Solution for the Heat Equation

1. The Fundamental solution

As we will see, in the case $\Omega = \mathbb{R}^n$, we will be able to represent general solutions the inhomogeneous heat equation

$$u_t - D\Delta u = f, \qquad \Delta \stackrel{\text{def}}{=} \sum_{i=1}^{n} \partial_i^2 \tag{1.0.1}$$

in terms of f, the initial data, and a single solution that has very special properties. This special solution is called the *fundamental solution*.

Remark 1.0.1. Note that when $\Omega = \mathbb{R}^n$, there are no finite boundary conditions to worry about. However, we do have to worry about "boundary conditions at ∞." Roughly speaking, this means that we have to assume something about the growth rate of the solution as $|x| \to \infty$.

Definition 1.0.1. The fundamental solution $\Gamma_D(t,x)$ to (1.0.1) is defined to be

$$\Gamma_D(t,x) \stackrel{\text{def}}{=} \frac{1}{(4\pi Dt)^{n/2}} e^{-\frac{|x|^2}{4Dt}}, \qquad t > 0, x \in \mathbb{R}^n, \tag{1.0.2}$$

where $x \stackrel{\text{def}}{=} (x^1, \cdots, x^n)$, $|x|^2 \stackrel{\text{def}}{=} \sum_{i=1}^n (x^i)^2$.

Let's check that $\Gamma_D(t,x)$ solves (1.0.1) when $f = 0$ in the next lemma.

Lemma 1.0.1. *$\Gamma_D(t,x)$ is a solution to the heat equation (1.0.1) with right-hand side $f \equiv 0$ whenever $x \in \mathbb{R}^n$ and $t > 0$.*

Proof. We compute that $\partial_t \Gamma_D(t,x) = \big(-\frac{2\pi Dn}{(4\pi Dt)^{n/2+1}} + \frac{1}{(4\pi Dt)^{n/2}} \frac{|x|^2}{4Dt^2} \big) e^{-\frac{|x|^2}{4Dt}}$. Also, we compute $\partial_i \Gamma_D(t,x) = -\frac{2\pi x^i}{(4\pi Dt)^{n/2+1}} e^{-\frac{|x|^2}{4Dt}}$ and $\partial_i^2 \Gamma_D(t,x) = \big(-\frac{2\pi}{(4\pi Dt)^{n/2+1}} + \frac{1}{4Dt} \frac{2\pi (x^i)^2}{(4\pi Dt)^{n/2+1}} \big) e^{-\frac{|x|^2}{4Dt}}$, $D\Delta \Gamma_D(t,x) = \big(-\frac{2\pi Dn}{(4\pi Dt)^{n/2+1}} + \frac{1}{4Dt} \frac{2\pi D|x|^2}{(4\pi Dt)^{n/2+1}} \big) e^{-\frac{|x|^2}{4Dt}}$. Lemma 1.0.1 now easily follows. □

Here are a few very important properties of $\Gamma_D(t,x)$.

Lemma 1.0.2. *$\Gamma_D(t,x)$ has the following properties:*

Recall that $\nabla\times$ is the curl operator, so that e.g. $\nabla \times \mathbf{B} = \big(\partial_y B_3 - \partial_z B_2, \partial_z B_1 - \partial_x B_3, \partial_x B_2 - \partial_y B_1\big)$. Let's look for steady-state solutions with $\partial_t \mathbf{E} = \partial_t \mathbf{B} \equiv 0$. Then equation (1.1.2) implies that

$$\nabla \times \mathbf{E} = 0, \tag{1.1.3}$$

so that by Poincaré's lemma on $\mathbb{R}^3$, there exists a scalar-valued function $\phi(x, y, z)$ such that

$$\mathbf{E}(x, y, z) = -\nabla\phi(x, y, z). \tag{1.1.4}$$

The function ϕ is called an *electric potential*. Plugging (1.1.4) into the second of (1.1.1), and using the identity $\nabla \cdot \nabla\phi = \Delta\phi$, we deduce that

$$\Delta\phi(x, y, z) = -\rho(x, y, z). \tag{1.1.5}$$

This is exactly the Poisson equation (0.0.3) with inhomogeneous term $f \stackrel{\text{def}}{=} -\rho$. Thus, Poisson's equation is at the heart of electrostatics.

1.2. **Connections to complex analysis.** Let $z = x + iy$ (where $x, y \in \mathbb{R}$) be a complex number, and let $f(z) = u(z) + iv(z)$ be a complex-valued function (where $u, v \in \mathbb{R}$). We recall that f is said to be differentiable at z_0 if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1.2.1}$$

exists. If the limit exists, we denote it by $f'(z_0)$.

A fundamental result of complex analysis is the following: f is differentiable at $z_0 = x_0 + iy_0 \simeq (x_0, y_0)$ if and only if the real and imaginary parts of f verify the *Cauchy-Riemann equations* at z_0:

$$u_x(x_0, y_0) = v_y(x_0, y_0), \tag{1.2.2}$$

$$u_y(x_0, y_0) = -v_x(x_0, y_0). \tag{1.2.3}$$

Differentiating (1.2.2) and using the symmetry of mixed partial derivatives (we are assuming here that $u(x, y)$ and $v(x, y)$ are C^2 near (x_0, y_0)), we have

$$\Delta u \stackrel{\text{def}}{=} u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0, \tag{1.2.4}$$

$$\Delta v \stackrel{\text{def}}{=} v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0. \tag{1.2.5}$$

Thus, *the real and imaginary parts of a complex-differentiable function are harmonic*!

2. Well-posed Problems

Much like in the case of the heat equation, we are interested in well-posed problems for the Laplace and Poisson equations. Recall that well-posed problems are problems that i) have a solution; ii) the solutions are unique; and iii) the solution varies continuously with the data.

Let $\Omega \subset \mathbb{R}^n$ be a domain with a Lipschitz boundary, and let $\hat{N}$ denote the unit outward normal vector to $\partial\Omega$. We consider the PDE

$$\Delta u(x) = f(x), \qquad x \in \Omega, \tag{2.0.1}$$

Proof. As we have previously discussed, $\Delta\Phi = \delta$. Also using (1.0.5), we compute that

$$\Delta_y\big(\Phi(x-y) - \phi(x,y)\big) = \Delta_y\Phi(x-y) - \Delta_y\phi(x,y) = \delta(y-x). \tag{1.0.7}$$

Therefore, $\Phi(x-y) - \phi(x,y)$ verifies equation (1.0.2).

Furthermore, using (1.0.6), we have that $\Phi(x-\sigma) - \phi(x,\sigma) = 0$ whenever $\sigma \in \partial\Omega$. Thus, $\Phi(x-y) - \phi(x,y)$ also verifies the boundary condition (1.0.3).

□

The following technical proposition will play a role later in the course when we derive representation formulas for solutions to (1.0.1) in terms of Green functions.

Proposition 1.0.2 (**Representation formula for** u)**.** *Let Φ be the fundamental solution for Δ in $\mathbb{R}^n$, and let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u \in C^2(\overline{\Omega})$. Then for every $x \in \Omega$, we have the following representation formula for $u(x)$:*

$$u(x) = \int_\Omega \Phi(x-y)\Delta_y u(y)\, d^n y - \underbrace{\int_{\partial\Omega} \Phi(x-\sigma)\nabla_{\hat{N}(\sigma)} u(\sigma)\, d\sigma}_{\text{single layer potential}} + \underbrace{\int_{\partial\Omega} u(\sigma)\nabla_{\hat{N}(\sigma)}\Phi(x-\sigma)\, d\sigma}_{\text{double layer potential}}. \tag{1.0.8}$$

Proof. We'll do the proof for $n = 3$, in which case $\Phi(x) = -\frac{1}{4\pi|x|}$. Let $B_\epsilon(x)$ be a ball of radius $\epsilon > 0$ centered at x, and let $\Omega_\epsilon \stackrel{\text{def}}{=} \Omega \backslash B_\epsilon(x)$. Note that $\partial\Omega_\epsilon = \partial\Omega \cup -\partial B_\epsilon(x)$. Using Green's identity, we compute that

$$\begin{aligned}\int_{\Omega_\epsilon} \frac{1}{|x-y|}\Delta u(y)\, d^3y &= \int_{\partial\Omega_\epsilon} \frac{1}{|x-\sigma|}\nabla_{\hat{N}} u(\sigma) - u(\sigma)\nabla_{\hat{N}}\Big(\frac{1}{|x-\sigma|}\Big)\, d\sigma \\ &= \int_{\partial\Omega} \frac{1}{|x-\sigma|}\nabla_{\hat{N}} u(\sigma)\, d\sigma - \int_{\partial\Omega} u(\sigma)\nabla_{\hat{N}}\Big(\frac{1}{|x-\sigma|}\Big)\, d\sigma \\ &\quad - \int_{\partial B_\epsilon(x)} \frac{1}{|x-\sigma|}\nabla_{\hat{N}} u(\sigma)\, d\sigma + \int_{\partial B_\epsilon(x)} u(\sigma)\nabla_{\hat{N}}\Big(\frac{1}{|x-\sigma|}\Big)\, d\sigma.\end{aligned} \tag{1.0.9}$$

In the last two integrals above, $\hat{N}(\sigma)$ denotes the radially outward unit normal to the boundary of the ball $B_\epsilon(x)$. This corresponds to the "opposite" choice of normal that appears in the standard formulation of Green's identity, but we have compensated by adjusting the signs in front of the integrals.

Let's symbolically write (1.0.9) as

$$L = R1 + R2 + R3 + R4. \tag{1.0.10}$$

Our goal is to show that as $\epsilon \downarrow 0$, the following limits are achieved:

- $L \to -4\pi \int_\Omega \Phi(x-y)\Delta_y u(y)\, d^3y$
- $R1 \to 4\pi \times$ single layer potential
- $R2 \to -4\pi \times$ double layer potential
- $R3 \to 0$
- $R4 \to -4\pi u(x)$.

is a relationship between the fluid variables. This relationship is often empirically determined. A commonly studied equation of state is

$$p = K\rho^{\gamma} \tag{2.0.2a}$$

where $\gamma > 1$ and $K > 0$ are constants. For future use, we note that under (2.0.2a), we have

$$\partial_x p = K\gamma\rho^{\gamma-1}\partial_x\rho, \tag{2.0.2b}$$

$$\partial_x^2 p = K\gamma\rho^{\gamma-1}\partial_x^2\rho + K\gamma(\gamma-1)\rho^{\gamma-2}(\partial_x\rho)^2, \tag{2.0.2c}$$

Also for future use, we differentiate (2.0.1a) with respect to t and (2.0.1b) with respect to x to deduce that

$$\partial_t^2\rho + \rho\partial_t\partial_x v + v\partial_t\partial_x\rho + \partial_t\rho\partial_x v + \partial_t v\partial_x\rho = 0, \tag{2.0.3a}$$

$$\rho\partial_t\partial_x v + v\partial_t\partial_x\rho + \partial_t\rho\partial_x v + \partial_t v\partial_x\rho + v^2\partial_x^2\rho + 2\rho v\partial_x^2 v + 4v\partial_x\rho\partial_x v = -\partial_x^2 p. \tag{2.0.3b}$$

The theory of acoustics is based on *linearizing* (i.e. throwing away the nonlinear terms) the equations (2.0.3a)-(2.0.3b) around the static solutions $\rho = \bar{\rho} = const > 0$, $v = 0$, $p = \bar{p} = const > 0$. These static solutions model a fluid at rest. Let's assume that we make a small perturbation of this solution, i.e., that v is small, and that

$$\rho = \bar{\rho} + \delta, \tag{2.0.4}$$

where $\delta(t, x)$ is a small function.

Using the expansion (2.0.4), we now throw away (with the help of (2.0.2c)) all of the quadratic and higher-order small terms from (2.0.3a)-(2.0.3b) to obtain the following **approximating** system (the quantities that are assumed to be small are v, δ, and all of their partial derivatives):

$$\partial_t^2\delta + \bar{\rho}\partial_t\partial_x v = 0, \tag{2.0.5a}$$

$$\bar{\rho}\partial_t\partial_x v = -K\gamma\bar{\rho}^{\gamma-1}\partial_x^2\delta. \tag{2.0.5b}$$

Comparing (2.0.5a) and (2.0.5b), we see that δ verifies the following **approximating** equation

$$-\partial_t^2\delta + K\gamma\bar{\rho}^{\gamma-1}\partial_x^2\delta = 0. \tag{2.0.6}$$

Equation (2.0.6) is a wave equation for the perturbation $\delta(t, x)$! It models the propagation of *sound waves*. This is the linear theory of acoustics! Note that the speed associated to the equation (2.0.6) depends on the background density $\bar{\rho}$:

$$c = \sqrt{K\gamma\bar{\rho}^{\gamma-1}}. \tag{2.0.7}$$

When $\gamma > 1$, higher background density $\Longrightarrow$ faster sound speed propagation.

Remark 2.0.1. For air under "normal" atmospheric conditions, $\gamma = 1.4$ is a pretty good model.

Proof. Using the **Representation formula for** $\widetilde{U}(t,r;x)$ corollary, the differentiability of $\widetilde{F}$, and the continuity of $\widetilde{G}$, we have that

$$\begin{aligned} u(t,x) &= \lim_{r\to 0^+} U(t,r;x) = \lim_{r\to 0^+} \frac{\widetilde{U}(t,r;x)}{r} \\ &= \lim_{r\to 0^+} \frac{\widetilde{F}(r+t;x) - \widetilde{F}(t-r;x)}{2r} + \frac{1}{2r}\int_{\rho=-r+t}^{\rho=r+t} \widetilde{G}(\rho;x)\,d\rho \\ &= \partial_t \widetilde{F}(t;x) + \widetilde{G}(t;x). \end{aligned} \tag{1.0.3}$$

The $\partial_t \widetilde{F}(t;x)$ term on the right-hand side of (1.0.3) arises from the definition of a partial derivative, while to derive the $\widetilde{G}(t;x)$ term, we applied the fundamental theorem of calculus (think about both of these claims own your own!). By the definition of $\widetilde{F}$ and $\widetilde{G}$ (see the **Spherical averages** Proposition), it therefore follows from (1.0.3) that

$$u(t,x) = \partial_t\Big(t\frac{1}{4\pi t^2}\int_{\partial B_t(x)} f(\sigma)\,d\sigma\Big) + t\frac{1}{4\pi t^2}\int_{\partial B_t(x)} g(\sigma)\,d\sigma. \tag{1.0.4}$$

Differentiating under the integral sign, using the chain rule relation $\partial_t[f(x+t\omega)] = (\nabla f)(x+t\omega)\cdot\omega = \nabla_{\hat{N}(x+t\omega)} f(x+t\omega)$ (where $\hat{N}$ is the unit outward normal to $\partial B_t(x)$), and recalling that $d\sigma = t^2 d\omega$ on $\partial B_t(x)$, we have that

$$\begin{aligned} t\partial_t\Big(\frac{1}{4\pi t^2}\int_{\partial B_t(x)} f(\sigma)\,d\sigma\Big) &= t\partial_t\Big(\frac{1}{4\pi}\int_{\partial B_1(0)} [f(x+t\omega)]\,d\omega\Big) = \frac{t}{4\pi}\int_{\partial B_1(0)} \partial_t[f(x+t\omega)]\,d\omega \\ &= \frac{t}{4\pi}\int_{\partial B_1(0)} \nabla_{\hat{N}(x+t\omega)} f(x+t\omega)\,d\omega \\ &\overset{\text{def}}{=} \frac{1}{4\pi t}\int_{\partial B_t(x)} \nabla_{\hat{N}(\sigma)} f(\sigma)\,d\sigma. \end{aligned} \tag{1.0.5}$$

Combining (1.0.4) and (1.0.5), we have that

$$u(t,x) = \frac{1}{4\pi t^2}\int_{\partial B_t(x)} f(\sigma)\,d\sigma + \frac{1}{4\pi t}\int_{\partial B_t(x)} \nabla_{\hat{N}(\sigma)} f(\sigma)\,d\sigma + \frac{1}{4\pi t}\int_{\partial B_t(x)} g(\sigma)\,d\sigma. \tag{1.0.6}$$

We have thus shown (1.0.2).

The fact that $u \in C^2([0,\infty)\times\mathbb{R}^3)$ follows from differentiating the integrals in the formula (1.0.2) and using the hypotheses on f and g.

□

Exercise 1.0.1. Show that (1.0.3) holds.

Exercise 1.0.2. Verify that $u \in C^2([0,\infty)\times\mathbb{R}^3)$, as was claimed at the end of the proof above.

The Linear Wave Equation: A Geometric Point of View

We will now derive some very important results for solutions to the linear wave equation. The results will exploit interplay between geometry and analysis. Many of the techniques that we will discuss play a central role in current PDE research.

Let $R \in [0,\infty]$ and let $t \in [0,R]$. Let X be the past-directed timelike vectorfield defined by $X^\mu = -\delta_0^\mu$ (where δ_α^β is the Kronecker delta), and let ${}^{(X)}J^\mu[\phi(t,y)]$ be the compatible current (1.0.12) *associated to X. Note that by* (1.0.8), ${}^{(X)}J^\mu[\phi(t,y)] = \frac{1}{2}|\nabla_{t,y}\phi(t,y)|^2 = \frac{1}{2}\sum_{\mu=0}^n(\partial_\mu\phi)^2 = \frac{1}{2}\{(\partial_t\phi)^2 + \sum_{i=1}^n(\partial_i\phi)^2\}$. *Let $p \in \mathbb{R}^n$ and let $B_R(p) \subset \mathbb{R}^3$ denotes the* solid *Euclidean ball of radius R centered at p. Define the energy $E[\phi](t)$ by*

$$E[\phi](t) \stackrel{def}{=} \int_{B_{R-t}(p)} \hat{N}_\mu {}^{(X)}J^\mu[\phi(t,y)]\, d^n y = \frac{1}{2}\int_{B_{R-t}(p)} |\nabla_{t,y}\phi(t,y)|^2\, d^n y, \tag{2.0.4}$$

where $\hat{N}_\mu = \delta_\mu^0$ (and therefore $\hat{N}^\mu = -\delta_0^\mu$) is the past-pointing unit normal covector to $\{t\}\times B_{R-t}(p) \subset \mathbb{R}^4$. Then

$$E[\phi](t) \le E[\phi](0). \tag{2.0.5}$$

Proof. The goal is to apply Theorem 1.1 to the solid truncated backwards light cone $\mathcal{C}_{t,p;R} \stackrel{\text{def}}{=} \{(\tau,y) \in [0,t]\times\mathbb{R}^n \mid |y-p| \le R-\tau\}$ and to make use of the dominant energy condition. It is easy to see that $\partial\mathcal{C}_{t,p;R} = \mathcal{B} \cup \mathcal{M}_{t,p;R} \cup \mathcal{T}$, where $\mathcal{B} \stackrel{\text{def}}{=} \{0\}\times B_R(p)$ is the flat base of the truncated cone, $\mathcal{T} \stackrel{\text{def}}{=} \{t\}\times B_{R-t}(p)$ is the flat top of the truncated cone, and $\mathcal{M}_{t,p;R} \stackrel{\text{def}}{=} \{(\tau,y) \in [0,t]\times\mathbb{R}^n \mid |y-p| = R-\tau\}$ is the mantle of the truncated cone.

By Theorem 1.1, we have that

$$E[\phi](t) - E[\phi](0) + F[\phi] = \int_{\mathcal{C}_{t,p;R}} \partial_\mu\Big({}^{(X)}J^\mu[\phi(\tau,y)]\Big)\, d\tau d^n y, \tag{2.0.6}$$

where

$$F[\phi] \stackrel{\text{def}}{=} \int_{\mathcal{M}_{t,p;R}} \hat{N}_\alpha {}^{(X)}J^\alpha[\phi(\sigma)]\, d\sigma \tag{2.0.7}$$

is the "flux" associated to $\mathcal{M}_{t,p;R}$. Since ϕ solves the wave equation (2.0.1), and since ${}^{(X)}\pi_{\mu\nu} = 0$, the identity (1.0.13) implies that the right-hand side of (2.0.6) is 0. Therefore,

$$E[\phi](t) - E[\phi](0) + F[\phi] = 0. \tag{2.0.8}$$

We claim that $F[\phi] \ge 0$. The energy inequality (2.0.5) would then follow from (2.0.8). The key observation for showing that $F[\phi] \ge 0$ is the following. Along the mantle $\mathcal{M}_{t,p;R}$, it is easy to see (draw the picture!) that $\hat{N}_\mu = \underline{L}_\mu$, where $\underline{L}$ is a past-directed null vector. Therefore, the integrand in (2.0.7) is equal to $T_{\alpha\beta}X^\alpha\underline{L}^\beta$, and since X is a past-directed timelike vector, the dominant energy condition (1.0.6) implies that $T_{\alpha\beta}X^\alpha\underline{L}^\beta \ge 0$. Therefore, $F[\phi] \ge 0$ as desired.

□

Theorem 2.1 can easily be used to prove the following **local** uniqueness result for solutions to the linear wave equation.

Corollary 2.0.1 (**Uniqueness**). *Suppose that two C^2 solutions ϕ_1 and ϕ_2 to* (2.0.1) *have the same initial data on $B_R(p) \subset \mathbb{R}^n$. Then the two solutions agree on the "solid backwards light cone" $\mathcal{C}_{p;R} \stackrel{def}{=} \{(\tau,y) \mid 0 \le \tau \le R, 0 \le |y-p| \le R-\tau\}$.*

MATH 3120 & 5120 COURSE NOTES - CLASS MEETING # 15

Math 3120 & 5120 Introduction to PDEs, Spring 2019 Professor: Jared Speck

Class Meeting # 15: The Fourier Transform on $\mathbb{R}^n$

1. Introduction to the Fourier Transform

Earlier in the course, we learned that periodic functions $f \in L^2([-1,1])$ (of period 2) can be represented using a Fourier series:

$$f(x) \text{“}=\text{”} \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x) + \sum_{m=1}^{\infty} b_m \sin(m\pi x). \tag{1.0.1}$$

The “ = ” sign above is interpreted in the sense of the convergence of the sequence of partial sums associated to the right-hand side in the $L^2([-1,1])$ norm. The coefficients a_m and b_m represent the “amount of the frequency m” that the function f contains. These coefficients were related to f itself by

$$a_0 = \int_{-1}^{1} f(x)\, dx, \tag{1.0.2a}$$

$$a_m = \int_{-1}^{1} f(x) \cos(m\pi x)\, dx, \qquad (m \geq 1), \tag{1.0.2b}$$

$$b_m = \int_{-1}^{1} f(x) \sin(m\pi x)\, dx, \qquad (m \geq 1). \tag{1.0.2c}$$

The *Fourier transform* is a “continuous” version of the formula (1.0.1) for functions defined on the whole space $\mathbb{R}^n$. Our goal is to write functions f defined on $\mathbb{R}^n$ as a superposition of different frequencies. However, instead of discrete frequencies m, we will need to use “continuous frequencies” ξ.

Definition 1.0.1 (Fourier Transform). Let $f \in L^1(\mathbb{R}^n)$, i.e., $\int_{\mathbb{R}^n} |f(x)|\, d^n x < \infty$. The Fourier transform of f is denoted by $\hat{f}$, and it is a new function of the frequency variable $\xi \in \mathbb{R}^n$. It is defined for each frequency ξ as follows:

$$\hat{f}(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}\, d^n x, \tag{1.0.3}$$

where $\cdot$ denotes the Euclidean dot product, i.e., if $x = (x^1, \cdots, x^n)$ and $\xi = (\xi^1, \cdots, \xi^n)$, then $\xi \cdot x \stackrel{\text{def}}{=} \sum_{j=1}^{n} \xi^j x^j$. In the above formula, recall that if is r is a real number, then $e^{ir} = \cos r + i \sin r$.

The formula (1.0.3) is analogous to the formulas (1.0.2a)-(1.0.2c). It provides the “amount of the frequency component” ξ that f contains. Later in the course, we will derive an analog of the representation formula (1.0.1).

These conditions are necessary and sufficient in order for the function given in (1.0.2) to solve (1.0.1) when $V = 0$. Note in particular that (1.0.6) shows that the speed of the plane wave solution *depends on* $|\xi|$, and in particular that larger $|\xi|'s$ lead to larger speeds. The dependence of the speed of the plane wave on ξ is known as *dispersion*, and (1.0.5) is known as the *dispersion relation* of Schrödinger's equation.

Dispersion plays a very important role in the analysis of certain PDEs, and in particular Schrödinger's equation. Heuristically, one sometimes imagines that a "typical" solution to a dispersive PDE is composed of many free waves, each moving at a different speed and/or spatial direction (at least when the dispersion relation is non-trivial). The dispersive nature of the PDE suggests that the different free wave components in the solution should separate from each other. As we will see (see e.g. Theorem 2.1), this heuristic argument is sometimes rigorously borne out, and separation can cause the overall amplitude of the solution to decay in time (often at a rate of t to some negative power).

2. The Fundamental Solution

We are now going to study the following global Cauchy problem for Schrödinger's equation:

$$i\partial_t \psi(t,x) + \frac{1}{2}\Delta\psi(t,x) = 0, \tag{2.0.1a}$$

$$\psi(0,x) = \phi(x). \tag{2.0.1b}$$

Let's start by momentarily forgetting about the initial data and instead trying to find the fundamental solution $K(t,x)$ to equation (2.0.1a). We will precisely define the fundamental solution below; it is analogous to the fundamental solution for the heat equation. As we will see, the techniques from Fourier analysis that we have previously developed will allow us to derive the fundamental solution with relative ease. To this end, we set $\psi(t,x) = K(t,x)$, take the spatial Fourier of equation (2.0.1a), and use the Fourier transform property $(\partial_{\vec{\alpha}} K)\hat{}(t,\xi) = (2\pi i\xi)^{\vec{\alpha}}\hat{K}(t,\xi)$ (and in particular $(\Delta K)\hat{}(t,\xi) = -4\pi^2|\xi|^2\hat{K}(t,\xi)$) to deduce the following ODE for $\hat{K}(t,\xi)$:

$$i\partial_t \hat{K}(t,\xi) - 2\pi^2|\xi|^2\hat{K}(t,\xi) = 0. \tag{2.0.2}$$

We rewrite (2.0.2) as

$$\partial_t \ln \hat{K}(t,\xi) = -2\pi^2 i|\xi|^2, \tag{2.0.3}$$

which can be easily integrated to give

$$\hat{K}(t,\xi) = Ce^{-2\pi^2 it|\xi|^2}, \tag{2.0.4}$$

where $C(\xi)$ is a constant that we have to calculate.

To calculate $C(\xi)$, we recall that we are ultimately trying to solve the following initial value problem for Schrödinger's equation:

$$i\partial_t \psi(t,x) + \frac{1}{2}\Delta\psi(t,x) = 0, \tag{2.0.5a}$$

$$\psi(0,x) = \phi(x). \tag{2.0.5b}$$

Since $K(t,x)$ is supposed to be the fundamental solution, we would like (in analogy with the results of our study of the heat equation) the solution to (2.0.5a) - (2.0.5b) to be of the form

$$\psi(t,x) = (K(t,\cdot) * \phi(\cdot))(x). \tag{2.0.6}$$

Note in particular that the tail (i.e., all but the first two terms) can be bounded by

$$\|A^2 + A^3 + A^4 + \cdots\| = \|A^2(I - A)^{-1}\| \leq C\|A\|^2 \tag{2.0.13}$$

if $\|A\|$ is sufficiently small, where $C > 0$ is a constant that depends on n. We now apply (2.0.12) and (2.0.13) to the matrix M defined in (2.0.6) (where $A_\nu^\mu \stackrel{\text{def}}{=} \epsilon \nabla_\nu Y^\mu$), thereby arriving at (2.0.9).

To derive (2.0.10), we first Taylor expand the determinant (viewed as a real-valued function of matrices) for sufficiently small $\|A\|$:

$$\det(I + A) = 1 + A_\alpha^\alpha + O(\|A\|^2) \tag{2.0.14}$$

In (2.0.14), we have written $O(\|A\|^2)$ to denote a term that can be bounded by $C\|A\|^2$, where $C > 0$ is some positive constant independent of (all sufficiently small) A. The expansion (2.0.10) now follows from (2.0.9) and (2.0.14). We remark that you will derive the expansion (2.0.14) in your homework in more detail.

□

We will now "define" how various fields and their derivatives transform under a change of coordinates. A full justification of these definitions can be found in books on tensor analysis or differential geometry.

Definition 2.0.1 (**Transformation properties of fields**)**.** Let $\phi(x)$ be a scalar-valued function, let $m(x)$ be an (invertible) metric (depending on x) with components $m_{\mu\nu}(x)$, and let $x \to \widetilde{x}$ be a spacetime diffeomorphism. Then upon changing coordinates $x \to \widetilde{x}$, these quantities transform as follows:

$$\widetilde{\phi}(\widetilde{x}) \stackrel{\text{def}}{=} \phi|_{(x \circ \widetilde{x})}, \tag{2.0.15a}$$

$$\widetilde{\nabla}_\mu \widetilde{\phi}(\widetilde{x}) \stackrel{\text{def}}{=} (M^{-1})_\mu^\alpha|_{(x \circ \widetilde{x})} \nabla_\alpha \phi|_{x \circ \widetilde{x}}, \tag{2.0.15b}$$

$$\widetilde{m}_{\mu\nu}(\widetilde{x}) \stackrel{\text{def}}{=} (M^{-1})_\mu^\alpha|_{(x \circ \widetilde{x})} (M^{-1})_\nu^\beta|_{(x \circ \widetilde{x})} m_{\alpha\beta}|_{(x \circ \widetilde{x})}, \tag{2.0.15c}$$

$$(\widetilde{m}^{-1})^{\mu\nu}(\widetilde{x}) \stackrel{\text{def}}{=} M_\alpha^\mu|_{(x \circ \widetilde{x})} M_\beta^\nu|_{(x \circ \widetilde{x})} (m^{-1})^{\alpha\beta}|_{(x \circ \widetilde{x})}. \tag{2.0.15d}$$

Above and throughout, we use the notation

$$\nabla_\mu \stackrel{\text{def}}{=} \frac{\partial}{\partial x^\mu}, \tag{2.0.16a}$$

$$\widetilde{\nabla}_\mu \stackrel{\text{def}}{=} \frac{\partial}{\partial \widetilde{x}^\mu}, \tag{2.0.16b}$$

$M_\nu^\mu \stackrel{\text{def}}{=} \frac{\partial \widetilde{x}^\mu}{\partial x^\nu}$ is the derivative matrix defined in (2.0.6), and $(M^{-1})_\nu^\mu = \frac{\partial x^\mu}{\partial \widetilde{x}^\nu}$ is its inverse. Furthermore, the notation $x \circ \widetilde{x}$ indicates that we are viewing x as a function of $\widetilde{x}$; this is possible since $x \to \widetilde{x}$ is a diffeomorphism.

Remark 2.0.2. (2.0.15a) simply says that the transformed function $\widetilde{\phi}$ takes the same value at the new coordinate $\widetilde{x}$ that ϕ takes at the old coordinate x. (2.0.15b) is really just the chain rule expressing $\frac{\partial}{\partial \widetilde{x}^\mu}$ in terms of $\frac{\partial}{\partial x^\mu}$. (2.0.15c) is the standard transformation law for *tensors* with two upstairs indices. These transformation laws generalize to other tensors in a straightforward fashion; the generalization can be found in books on tensor analysis/ differential geometry. Roughly speaking, tensors with indices downstairs transform by multiplication by the matrix M^{-1} (one copy of M^{-1} for each index), and tensors with indices upstairs transform by multiplication by the matrix M (one copy of M for each index).