

Summary of Chapter 5

- The only generally soluble second-order equations are *linear differential equations with constant coefficients*,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x).$$

If $f(x) = 0$, the equation is *homogeneous*. If $f(x) \neq 0$ it is *inhomogeneous*.

- The solutions of *homogeneous* equations all take the form e^{mx} where m may be real, imaginary or complex. The values of m are the roots of the auxiliary equation

$$m^2 + am + b = 0.$$

- If $a^2 > 4b$, the roots m_1 and m_2 are real and distinct so that we can write

$$y = Ce^{m_1x} + De^{m_2x}.$$

If $a^2 < 4b$, the roots are complex conjugates $m = -a/2 \pm ik$ so that we can write

$$\begin{aligned} y &= e^{-ax/2} (Ce^{ikx} + De^{-ikx}) \\ &= e^{-ax/2} (E \cos kx + F \sin kx). \end{aligned}$$

If $a^2 = 4b$, the repeated root is real, $m = -a/2$. The solution is then

$$y = Gxe^{mx} + He^{mx}.$$

- When $a = 0$ and $b > 0$, we have *simple harmonic motion*. When $a > 0$ and $b > 0$, we have a *damped harmonic oscillator*.

Summary of Chapter 6

- The general solution of inhomogeneous linear differential equations can be written as

$$y(x) = y_p(x) + y_h(x),$$

where $y_h(x)$ is the solution of the corresponding homogeneous equation and $y_p(x)$ is a particular solution of the inhomogeneous equation.

- An example of second-order inhomogeneous linear differential equation is the harmonic oscillator with periodic forcing

$$\ddot{y} + \omega_0^2 y = f \cos(\omega t).$$

For $\omega \approx \omega_0$ the resulting oscillations show very large amplitude, a phenomenon known as **resonance**.

- A system of *two coupled* equations is written as

$$\frac{dx}{dt} = f(x, y, t), \quad \frac{dy}{dt} = g(x, y, t),$$

and is generally intractable.

- In simple cases, one of the equations is independent of the other. The solution of this equation can then be introduced into the second. Sometimes a *substitution* for x and y may produce independent equations.
- In cases of *linear equations with constant coefficients*,

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy,$$

one variable may be eliminated by first *differentiating* one equation,

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt}$$

then substituting for dy/dt using the other equation,

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + dy),$$

and finally eliminating y by using the first equation again,

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + bcx + bd \left(\frac{dx/dt - ax}{b} \right).$$

This produces a *second-order differential equation with constant coefficients*.