Lec 1 – Risk Aversion & Returns

Return

Definitions

$$HPR_{01} \equiv \frac{P_1 + D_1 - P_0}{P_0}$$

Unknown future return: $\tilde{r}$ Expected return: $E(\tilde{r})$ Realised return: r

$Tilde$: ~ denotes RV

$$E(\tilde{r}) = \sum_{s=1}^{n} \Pr(Scenario) \times Possible\ Return = \sum_{s=1}^{n} p_s \times \tilde{r}_s$$

$$Var(\tilde{r}) = \sum_{s=1}^{n} \Pr(Scenario) \times \left(Possible\ Return - E(r)\right)^2 = \sum_{s=1}^{n} p_s \times \left(r_s - E(r)\right)^2$$

$$= E[\left(\tilde{r}_s - E(\tilde{r})\right)^2]$$

Modelling: Normal Distribution

Advantages

Tractable: symmetric, and only 2 parameters needed (so σ is an appropriate measure of risk)

Stable: if asset returns ~ N, then portfolio returns will also be ~ N

Empirically a reasonable first approx.

Easy to compute conf intervals: $\Pr(\tilde{r}_t \leq x) = \Pr(\tilde{z} \leq \frac{x-\mu}{\sigma})$

Disadvantages

While log returns ($\ln(\frac{P_1+D_1}{P_0})$) are close to ~N, they have fat tails (too many outliers), thus requiring a t-distribution w/ certain no. of df

Skew

$\frac{E(\tilde{r}-\mu)^3}{\sigma^3}$ Pos. skew = extremes to right; downside risk is overestimated by σ.

Kurtosis

Tail fatness Kurtosis = $\frac{E[(\tilde{r}-\mu)^4]}{\sigma^4}$ (-3 for excess kurtosis)

Platykurtic (neg) vs Leptokurtic (pos): flatter vs taller

Portfolio Theory

Concept

Portfolio Definition

Any collection of investments (assets)

Risk Ctrl

2 Methods:

- Cap Allocation: shift funds b/w risky & risk-free assets
- Efficient Diversification: shift funds b/w risky assets

Both have 2 elements:

- Objective element: risk-return combos available
- Subjective element: risk-return preferences of investor

Cap Allocation

Splitting funds b/w safe & risky assets

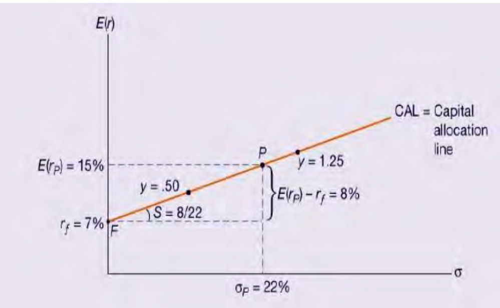
Feasible Portfolios

$E(\tilde{r}_c) = wE(\tilde{r}_p) + (1-w)r_f = r_f + w\underbrace{[E(\tilde{r}_p) - r_f]}_{\text{Risk Premium}}$ $\quad w = weight\ on\ risky\ asset$

$\sigma_c^2 = w^2\sigma_p^2$ (Var of rf asset = 0)

CAL (Cap Allocation Line)

Delineates set of feasible portfolios resulting fm diff values of w



$$E(r_c) = r_f + \sigma_c \underbrace{\frac{E(\tilde{r}_p) - r_f}{\sigma_p}}$$

Sharpe ratio = gradient:

AKA reward-volatility ratio

Special case: **Cap Mkt Line (CML):** P = M (broad mkt index, i.e. mkt portfolio)

Utility

Mean Variance Utility Function

$U = E(r) - \frac{1}{2}A\sigma^2$ U = Utility

A = risk attitude (>/=/< 0 for risk averse/neutral/seeking)

Optimal Portfolios

Use Mean Variance Utility Function to choose optimal portfolio fm set of feasible portfolios

$$w^* = \frac{E(r_p) - r_f}{A\sigma_p^2}$$

Lec 2 Optimal Portfolio Choice

MAXIMUM SHARPE RATIO WHEN GIVEN 2 RISKY SECURITIES (B, S)

$$w_B = \frac{[E(r_B) - r_f]\sigma_S^2 - [E(r_s) - r_f]\sigma_{B,S}}{[E(r_B) - r_f]\sigma_S^2 + [E(r_S) - r_f]\sigma_B^2 - [E(r_B) - r_f + E(r_S) - r_f]\sigma_{B,S}}$$

Portfolio Construction Process

1. Cap Allocation (Lec 1)

How much in risky assets & how much in rf?

- Depends on what's available (CAL) & investor's risk-return prefs (utility function)

2. Efficient Diversification

Choose optimal risky portfolio: which risky assets & how much in each?

Portfolio Return & Risk

Definitions

$\tilde{r}_p(s) = \sum_i w_i \tilde{r}_i(s)$ $\quad\quad w_i = weight\ in\ asset\ i: \frac{W_i}{W}$

$W_i = \$amt\ in\ asset\ i$ $\quad\quad W_i = Portfolio\ value$

$Portfolio\ realised\ return: r_P = \sum_i w_i r_i$ $\quad\quad Portfolio\ E(r): r_P = \sum_i w_i (\tilde{r}_i)$

$\sigma_p^2 = \sum_{j=1}^m \sum_{k=1}^m w_j w_k \sigma_{jk}$ = $w^2\sigma^2$ for each security + $2w_Aw_B\sigma_{AB}$ for each combo of securities

$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \times \sigma_j}$ $\quad\quad \sigma_{ij} = \sum_s p(s)[\tilde{r}_i(s) - E(\tilde{r}_i)] \times [\tilde{r}_j(s) - E(\tilde{r}_j)]$

Diversification benefit

$\boldsymbol{\rho_{12} = +1}$ No diversification benefit

$\boldsymbol{\rho_{12} = -1}$ Max diversification benefit

- Zero risk portfolio: $\sigma_p = 0$

  $\sigma_p = w_1\sigma_1 - (1 - w_1)\sigma_2 = 0$ and solve for w_1: $\quad w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}$

  If avg cov across many assets = 0, portfolio risk can be driven to 0 as no. of assets in portfolio increases

$\boldsymbol{-1 < \rho_{12} < +1}$ Most realistic; some diversification benefit always exist

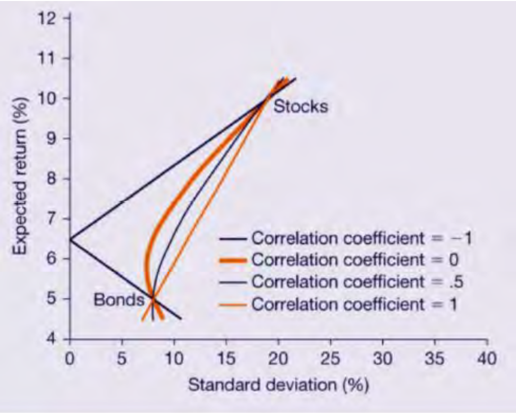
- Min. risk: $\quad w_1 = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$

  If avg cov across many assets >0, portfolio risk remains >0 as no. of assets in portfolio increases. However, <u>portfolio risk will still always decrease as no. of assets increases.</u>

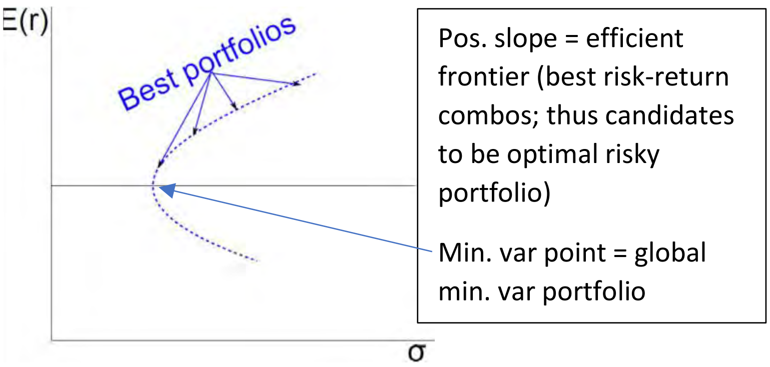
Efficient diversification

Achieve lowest portfolio risk at any given level of E(r).

Minimise: $\sigma_p^2 = \sum_{j=1}^m \sum_{k=1}^m w_j w_k \sigma_{jk}$ subject to: $E(r_p) = \sum_{i=1}^n w_i E(r_i) = r^*$ and $\sum_{i=1}^n w_i = 1$



Min var frontier (lowest risk for each r*)



Practical Restraints

Each new constraint shrinks investment opportunity set

- No short (no neg w)
- Limit allowed volatility (portfolio sd not to exceed x)
- Limit benchmarking tracking error: investors may need investment manager not to depart 'too far' fm an investment style
- Limit exposure to certain investment classes

Adding RF asset

Best possible CAL: find risky portfolio w/ highest Sh ratio tangential to efficient frontier

Maximise: $\frac{E(\tilde{r}_p)-r_f}{\sigma_p}$ subject to: $E(r_p) = \sum_{i=1}^{n} w_i E(r_i)$; $\sigma_p^2 = \sum_{j=1}^{m}\sum_{k=1}^{m} w_j w_k \sigma_{jk}$; $\sum_{i=1}^{n} w_i = 1$

Line to right of T = borrow at rf; line to left of T = lend at rf

2 Fund Separation

Choose optimal portfolio

1. Find optimal risky portfolio T (same for all ppl)
2. Find optimal complete portfolio consisting of T & rf asset
   Varies b/w investors depending on their risk-return prefs (utility function): investors w/ high risk aversion will put more in rf & less in T

Implications

Can invest in T on behalf of all clients, then tailor it to individual clients by combining T w/ different weights in rf: low weight for low risk aversion

Lec 3 CAPM

Asset Pricing

Concept

Major determinant of E(r) = risk. Risk ∝ E(r).

Portfolio Theory vs Asset Pricing

Portfolio theory: how should **someone** invest given their conditions

Asset pricing: how will **mkt** determine E(r) of assets available.

CAPM: $E(r_i) = r_f + [E(r_M) - r_f]\beta_i$

Assumptions

1. Competitive mkts; individual investors are price-takers, i.e. security prices aren't affected by individual traders.
2. All investors have same single-period planning horizon.
3. Investments limited to:
   - Publicly traded financial assets
   - RF borrowing & lending available to all investors
4. No mkt frictions, e.g. taxes & transaction costs.
5. All investors are rational mean-variance optimisers: they maximise $U = E(r) - \frac{1}{2}A\sigma^2$
6. Info's costless & available to all investors
7. Homogenous expectations: all investors interpret info identically.
   - Everyone sees same μ & σ^2 for same set of assets; everyone draws same risky frontier & invests in same tangency portfolio *T*.

Implications

T=M

Tangency portfolio *T* = mkt portfolio *M* (weighted sum of all risky assets)

There r *H* investors (*h* = 1, … , *H*). Each investor (*h*) invests in *T* & combines it w/ a position (pos/neg) in rf. Proportion in *T* = y^h

Aggregate wealth $P_T = \sum_{h=1}^{H} W^h = \sum_{h=1}^{H} y^h W^h + \sum_{h=1}^{H}(1 - y^h)W^h$

Total borrowing = total lending: $\sum_{h=1}^{H}(1 - y^h)W^h = 0 \therefore P_T = \sum_{h=1}^{H} y^h W^h$

Asset supply: weight of asset *i* in portfolio *M* = $w_{i,m} = \frac{P_i}{P_M}$

Total asset supply = mkt value of mkt portfolio, i.e. P_M

For any asset *I*, supply = demand: $P_i = \sum_{h=1}^{H} w_i^h y^h W^h$

Every investor holds *T*: $w_i^h = w_{i,T}\ for\ all\ h$

$\therefore P_i = \sum_{h=1}^{H} w_{i,T} y^h W^h = w_{i,T} \sum_{h=1}^{H} y^h W^h = w_{i,T} P_T$ $\qquad \therefore w_{i,T} = \frac{P_i}{P_T}$

Aggregate supply = aggregate demand: $P_M = P_T$ $\qquad \therefore w_{i,T} = \frac{P_i}{P_M} = w_{i,M}$