Investment Analysis (25503) Notes

Lecture 1: Introduction, Returns, Basic Statistics and Matrices

2 types of securities:

1. **Fixed income securities** – government bonds, treasury bills, treasury notes, convertible bonds, commercial paper
2. **Variable income securities** – preferred stock, common stock, investment companies, mutual funds

Harry Markowitz (1952) – diversification from a mathematical point of view

- How to construct a portfolio to have the **greatest possible return for its given level of risk**
- Optimal asset allocation requires quantitative skills in mathematics and statistics

Equilibrium pricing models

After Markowitz, 2 pricing models were developed that describe how risk assets are priced in equilibrium:

1. **Capital Asset Pricing Model (CAPM)** – developed by Sharpe, Linter and Mossin
2. **Arbitrage Price Theory (APT)** – developed by Ross

Simple annual return (no interest on interest)

$$F = P(1+rt) \quad \text{implies:} \quad r = \frac{1}{t}\left(\frac{F}{P} - 1\right).$$

Compound annual return

$$F = P\left(1+\frac{r}{n}\right)^{nt} \quad \text{implies:} \quad r = n\left[\left(\frac{F}{P}\right)^{1/(nt)} - 1\right].$$

Effective annual rate of return – return that would have been observed if compounding had occurred annually → related to nominal annual rate of return

$$1 + r_e = \left(1+\frac{r}{n}\right)^n.$$

Continuously compounded annual return

$$F = P\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = P\,e^{rt} \quad \text{implies:} \quad r = \frac{1}{t}\ln\left(\frac{F}{P}\right).$$

Note: annual rate decreases as compounding period becomes smaller

Discrete random variable – can take only countably many values → finite outcome (e.g. die)
Continuous random variable – takes value in an interval or over the whole real line → infinite value b/w 2 numbers

Population mean

⋆ For a discrete random variable X, the population mean, μ_X, is

$$\mu_X = \mathrm{E}[X] = \sum_{i=1}^{M} p_i x_i.$$

Population variance

⋆ For a discrete random variable X, the population variance, σ_X^2, is

$$\sigma_X^2 = \mathrm{Var}(X) = \mathrm{E}\left[(X-\mathrm{E}[X])^2\right] = \sum_{i=1}^{M} p_i\,(x_i - \mathrm{E}[X])^2.$$

variance = $\sum(deviation - mean)^2$

Population covariance

⋆ The population covariance, $\sigma_{X,Y}$, for two random variables X and Y is

$$\sigma_{X,Y} = \mathrm{Cov}(X,Y) = \mathrm{E}\left[(X-\mathrm{E}[X])\,(Y-\mathrm{E}[Y])\right].$$

Population correlation coefficient

⋆ The population correlation coefficient, $\rho_{X,Y}$, for two random variables X and Y is

$$\rho_{X,Y} = \frac{\sigma_{X,Y}}{\sigma_X \sigma_Y}.$$

covariance = (SD of X – mean)(SD of Y – mean)

CC = $\frac{Covariance}{SD\ of\ x \times SD\ of\ y}$

Sample mean

⋆ The sample mean, $\hat{\mu}_X$, of a random variable X, based on the observations $x_1, \ldots, x_N$, is

$$\hat{\mu}_X = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

Sample covariance

⋆ The sample covariance, $\hat{\sigma}_{X,Y}$, of the random variables X and Y, based on joint observations $(x_1, y_1), \ldots, (x_N, y_N)$, is

$$\hat{\sigma}_{X,Y} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu}_X)(y_i - \hat{\mu}_Y).$$

Sample variance

⋆ The sample variance, $\hat{\sigma}_X^2$, of a random variable X, based on the observations $x_1, \ldots, x_N$, is

$$\hat{\sigma}_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu}_X)^2.$$

Sample correlation coefficient

⋆ The sample correlation coefficient, $\hat{\rho}_{X,Y}$, for two random variables X and Y based on joint observations $(x_1, y_1), \ldots, (x_N, y_N)$, is

$$\hat{\rho}_{X,Y} = \frac{\hat{\sigma}_{X,Y}}{\hat{\sigma}_X \hat{\sigma}_Y}.$$

variance = $\sum(observation - mean)^2$

Since portfolios are linear combination of assets

$$Z = aX + bY,$$

$$\mu_Z = a\mu_X + b\mu_Y$$

$$\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{X,Y}.$$

A number of operations can be performed on matrices:

- **Matrix transposition** - Transposition, denoted ‘T’, swaps the rows and columns of a matrix.
- **Matrix addition and subtraction** – only if dimensions agree
- **Matrix multiplication** – multiplied by scalars α
  - Only if # of columns in left-hand matrix = # of rows in right-hand matrix.

The variance-covariance matrix is

$$\Omega = \begin{pmatrix} \text{Var}(r_A) & \text{Cov}(r_A, r_B) \\ \text{Cov}(r_B, r_A) & \text{Var}(r_B) \end{pmatrix}$$

Diagonal matrix – n*n matrix with zeroes everywhere, except on its diagonal

Invertible matrix –

Lecture 2: Individual Choice and Two-Asset Portfolios

Hence your expected return

$$\mu_P = E(r_P) = x_1E(r_1) + x_2E(r_2) = x_1\mu_1 + x_2\mu_2$$

The variance of your portfolio is

$$\sigma_P^2 = Var(r_P) = Var(x_1r_1 + x_2r_2) = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{1,2}$$

Where x1 and x2 are the weighted averages

A **portfolio** is a collection of assets held by an investor. **Portfolio weights** define the fractional amount of money invested in each of the assets in the portfolio (x_i)

Budget constraint – condition whereby portfolio weights defined by the fractional amount invested in each asset must sum to 1

- $x_1 + x_2 = 1$ or in matrix form: $\boldsymbol{x}^\top \mathbf{1} = 1.$

Short selling is achieved by borrowing the security from someone and then selling it in the market

- When $x_i > 0$, the portfolio contains a **long position**
- When $x_i < 0$ (i.e. NEGATIVE WEIGHT), the portfolio contains a **short position**
- A portfolio **without short selling** satisfies the inequalities $0 \leq x_i \leq 1$

Risk and return of a portfolio

The return of a two-asset portfolio, r_P, is $r_P = x_1 r_1 + x_2 r_2$. The expected or mean return of a two-asset portfolio, $\mu_P = E(r_P)$, is $\mu_P = x_1\mu_1 + x_2\mu_2$. In matrix form:

$$r_P = \boldsymbol{x}^\top \boldsymbol{r} \quad \text{and} \quad \mu_P = \boldsymbol{x}^\top \boldsymbol{\mu}.$$

The variance of returns for a two-asset portfolio, σ_P^2, is $\sigma_P^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{1,2}$, or in matrix form:

$$\sigma_P^2 = \boldsymbol{x}^\top \Omega \boldsymbol{x}.$$

A **combination line** is the set of points in mean-standard deviation space that are achievable by combining together two assets into one portfolio

- Generally a **hyperbola** in mean-SD space but can be a **straight line** in special cases
- Tells you what the relationship between expected return and standard deviation is for any portfolio constructed from the two securities.

Return and risk for an arbitrary portfolio with fraction invested in asset 1

$$\mu_P = \boldsymbol{x}^\top \boldsymbol{\mu} = x_1\mu_1 + (1-x_1)\mu_2,$$

$$\sigma_P^2 = \boldsymbol{x}^\top \Omega \boldsymbol{x} = x_1^2\sigma_1^2 + (1-x_1)^2\sigma_2^2 + 2x_1(1-x_1)\rho_{1,2}\sigma_1\sigma_2.$$

Global Minimum Variance Portfolio (GMVP) – portfolio with the least possible variance of returns

- *Derive = 0* $\frac{d\sigma_P^2}{dx_1} = 0$ → therefore weight of asset 1 → $x_1^* = \frac{\sigma_2^2 - \sigma_{1,2}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{1,2}}.$

Portfolio of 1 risk-free asset and 1 risky asset

$$\mu_P = r_F \pm \frac{\mu_1 - r_F}{\sigma_1}\sigma_P.$$

Combination line becomes a *straight line*

- When $x_1 > 0$, you are **leveraging** (borrowing at risk-free rate and investing proceeds in risky security)
- When $x_1 < 0$ you are **lending** at the risk free rate

Individual Choice

We assume individuals make investment decisions that **maximise their utility** (happiness). When outcome is uncertain, they maximise their *expected* utility.

- Assume utility, U, is only a function of wealth and that investors prefer more wealth to less wealth → **non-satiated** meaning utility is increasing → U'(w) > 0
- **Marginal utility is diminishing** as wealth increases – investors value each additional unit of wealth less than previous unit of wealth → U''(w) < 0

Risk of asset can measured by SD of its return as long as one of following 2 conditions hold:

1. **Asset returns are jointly normally distributed**
2. **Utility functions are quadratic U(W) = aW – bW^2**

Efficient assets

- If asset A exists in upper-left quadrant from asset B, then asset A is said to **dominate** asset B. Also asset A is said to be **inefficient**
- If no other assets exist in upper-left quadrant from asset A, then asset A is said to be **efficient**
- If several assets are efficient, investor chooses asset that maximizes expected utility

Indifference curves – collection of points over which expected utility is the same.

- Determines graphically which asset maximises investor's utility
- The top left indifference curve is always the most desired
- Optimal portfolio for choosing b/w efficient portfolios is the **point on the hyperbola that is tangent to the investor's indifference curve**

Lecture 3: Mean-Variance Portfolio Theory

First define the Lagrangian:

$$L(x_1, x_2, x_3, \lambda, \gamma) = \frac{1}{2}\boldsymbol{x}^\top \Omega \boldsymbol{x} + \lambda[1 - \boldsymbol{x}^T \mathbf{1}] + \gamma[\mu_P - \boldsymbol{x}^\top \boldsymbol{\mu}]$$

3.2.1 The Markowitz problem

⋆ Formally, for a given fixed expected rate of return μ_P, we wish to find the vector of portfolio weights $\boldsymbol{x}^*$ that minimize the variance. The minimization problem is

$$\boldsymbol{x}^{*\top} \Omega \boldsymbol{x}^* = \min_{\boldsymbol{x}} \boldsymbol{x}^\top \Omega \boldsymbol{x}, \quad (3.8)$$

subject to

$$\text{the budget constraint} \quad \boldsymbol{x}^\top \mathbf{1} = 1, \quad (3.9)$$
$$\text{the target return constraint} \quad \boldsymbol{x}^\top \boldsymbol{\mu} = \mu_P. \quad (3.10)$$

⋆ The constrained minimization problem (3.8)–(3.10) is called the Markowitz problem.

The corresponding solution is called **optimal portfolio** or **Mean-Variance frontier portfolio**

The first-order conditions for the optimal weights, $\boldsymbol{x}^*$, and Lagrange multipliers, λ and γ, are (see the previous page):

$$\Omega \boldsymbol{x}^* = \lambda \mathbf{1} + \gamma \boldsymbol{\mu}, \quad (3.11)$$
$$\mathbf{1}^\top \boldsymbol{x}^* = 1, \quad (3.12)$$
$$\boldsymbol{\mu}^\top \boldsymbol{x}^* = \mu_P. \quad (3.13)$$

For convenience, we define the following scalars:

$$A = \mathbf{1}^\top \Omega^{-1} \mathbf{1}, \quad B = \mathbf{1}^\top \Omega^{-1} \boldsymbol{\mu}, \quad C = \boldsymbol{\mu}^\top \Omega^{-1} \boldsymbol{\mu}, \quad \Delta = AC - B^2.$$

$$\lambda = \frac{C - B\mu_P}{\Delta}, \quad \gamma = \frac{A\mu_P - B}{\Delta}, \quad (3.17)$$

$$\boldsymbol{x}^* = \lambda \Omega^{-1} \mathbf{1} + \gamma \Omega^{-1} \boldsymbol{\mu}. \quad (3.18)$$

Minimum Variance Set (MVS) – all portfolios that have minimum variance for a certain target portfolio returns:

- To determine MVS (and thus efficient frontier) → set target portfolio and determine which portfolio with this return has the minimum variance
- Hyperbolic curve that bounds region of possible portfolios

Variance of Optimal portfolio:

$$\sigma_P^2 = \frac{A\mu_P^2 - 2B\mu_P + C}{\Delta}$$

When 2 assets are combined, resulting portfolios exist along hyperbolic curve in mean-SD space

- Portfolios along upper half of MVS starting at MVP are efficient → this section of MVS is known as the **efficient frontier**
- Lower half of MVS is known as the **inefficient frontier**

Asymptotes of the MVS

⋆ Since the MVS is a hyperbola in mean-standard deviation space, it possesses asymptotes:

$$\mu_P = \frac{B}{A} \pm \sqrt{\frac{\Delta}{A}} \sigma_P.$$

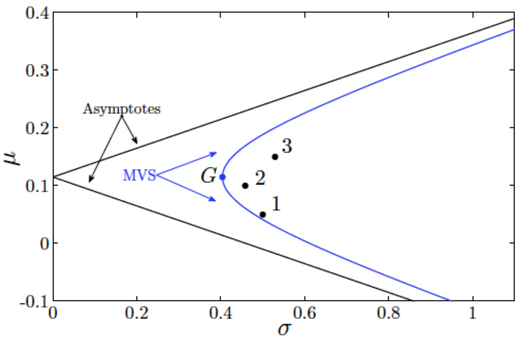
Global MVP

⋆ To locate the global minimum variance portfolio, G, we first compute the derivative of σ_P^2 with respect to μ_P using (3.19),

$$\frac{d\sigma_P^2}{d\mu_P} = 2\frac{A\mu_P - B}{\Delta}.$$

Setting this equal to zero yields the expected return of the MVP,

$$\mu_G = \frac{B}{A}.$$



Variance of returns of MVP:

$$\sigma_G^2 = \frac{1}{A}.$$

Weights of the MVP:

$$\boldsymbol{x}_G = \frac{\Omega^{-1}\mathbf{1}}{A} = \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^\top\Omega^{-1}\mathbf{1}}.$$

Two Fund Theorems

The **Two Fund Theorem** states the entire MVS can be created by combining together any 2 distinct portfolios that already exist on the MVS → to achieve desired (Er) of investor

- Result suggests that investment mgmt company only needs to create 2 funds for all of their clients → to satisfy particular client's risk preference, just need to mix 2 funds together to reach specific point on efficient frontier that the client desires
- When there is a risk-free asset → only 1 fund of risky assets is needed to generate entire MVS

Lecture 4: Mean-Variance Portfolio Theory 2: Risky assets and Risk-free asset

Determining MVS graphically → when adding a risk-free asset to a collection of risky assets, we can generate the set of possible portfolios via a 2-step process

1. Determine set of possible portfolios that can be constructed using only risky assets → generally be a shaded hyperbolic region in mean-SD space
2. Draw combination lines that connect risk-free security to each of possible risky portfolios. When aggregated together, lines form a solid triangular region in mean-SD space
   - **MVS** is the outer boundary of the triangle. It is the combination line with the highest slope → line that connects risk-free asset to point that is just tangent to risky security MVS**.** The portfolio at this tangency point is called the **tangency portfolio (T)**

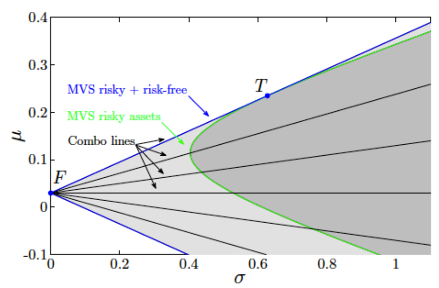


Figure 4.1: The MVS (shown in blue) for the three risky assets from Lecture 3 plus a risk-free asset with $r_F = 0.03$. Several combination lines that connect the risk-free asset with various risky portfolios are shown. The MVS is the outer region of the shaded triangle and is defined by the combination line that connects F with the tangency portfolio, T.

Budget constraint

⋆ The new budget constraint is

$$\boldsymbol{x}^\top\mathbf{1} + x_0 = 1 \quad \Longleftrightarrow \quad x_0 = 1 - \boldsymbol{x}^\top\mathbf{1}$$

Risk and return of the total portfolio

⋆ The return and expected return of the portfolio are:

$$r_P = \boldsymbol{x}^\top\boldsymbol{r} + x_0 r_F \quad \text{and} \quad \mu_P = \boldsymbol{x}^\top\boldsymbol{\mu} + [1 - \boldsymbol{x}^\top\mathbf{1}]r_F.$$

⋆ The risk-free asset does not add risk to the portfolio, so the variance equation does not change:

$$\sigma_P^2 = \boldsymbol{x}^\top\Omega\boldsymbol{x}.$$