

Functions of two and three variables.

1. review: When the derivate wrt one variable is taken, the other variables are treated as constants.

e.g. $f(x, y) = \cos(2x) - x^2 e^{5y} + 3y^2$

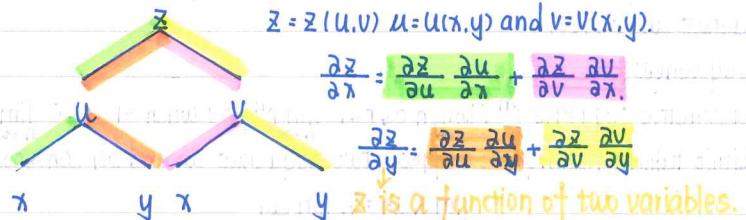
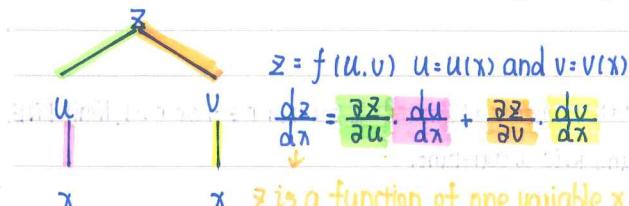
$$f_x = -2\sin(2x) - 2x \cdot e^{5y} \quad f_y = -5x^2 e^{5y} + 6y$$

$$f_{xx} = -4\cos(2x) - 2e^{5y} \quad f_{yy} = -25x^2 e^{5y} + 6$$

$$f_{xy} = f_{yx} = -10e^{5y}$$

Note that in this course $f_{xy} = f_{yx}$.

2. the chain rule: To find the rate of change of the dependent variable wrt the independent variable, travel down all paths from the dependent variable to the independent variable, by taking partial derivatives or derivatives, multiply all derivatives / partial derivatives along each path and add these products.



3. implicit differentiation.

Given function: $F(x, y) = 0$. $\frac{dy}{dx} = -\frac{F_x}{F_y}$, $\frac{dx}{dy} = -\frac{F_y}{F_x}$.

Given function: $F(x, y, z) = 0$. $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$, $\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$ and $\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$.

4. gradient and directional derivatives.

The gradient of the function $f(x, y)$ at (x_0, y_0) is $\nabla f(x_0, y_0) = (f_x, f_y)|_{(x_0, y_0)}$.

The gradient of the function $f(x, y, z)$ at (x_0, y_0, z_0) is $\nabla f(x_0, y_0, z_0) = (f_x, f_y, f_z)|_{(x_0, y_0, z_0)}$.

e.g. Given a surface $z = x^2 - y$. Find $\nabla z(1, 2)$.

$$\frac{\partial z}{\partial x} = 2x, \frac{\partial z}{\partial y} = -1 \quad \nabla z(1, 2) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right)|_{(1, 2)} = (2, -1)|_{(1, 2)} = (2, -1).$$

If \vec{u} is a vector tangent to the level curve f at (x_0, y_0) then $\nabla f(x_0, y_0) \cdot \vec{u} = 0$.

If \vec{u} is any unit vector, the directional derivative of f at (x_0, y_0) in the direction of \vec{u} , given by $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$ measures the rate of change of f at (x_0, y_0) as (x, y) moves from (x_0, y_0) in the direction of \vec{u} .

e.g. Find the directional derivative of $f(x, y) = 5 - x^2 + y^2$ at the point $(1, 1)$ in the direction of $(3, -4)$.

$$\| (3, -4) \| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\vec{u} = \left(\frac{3}{5}, -\frac{4}{5} \right) \quad \nabla f|_{(1, 1)} = (f_x, f_y)|_{(1, 1)} = (-2x, 2y)|_{(1, 1)} = (-2, 2)$$

$$D_{\vec{u}} f(1, 1) = (-2, 2) \cdot \left(\frac{3}{5}, -\frac{4}{5} \right) = -2 \cdot \frac{3}{5} + 2 \cdot -\frac{4}{5} = -\frac{6}{5} - \frac{8}{5} = -\frac{14}{5}.$$

Because $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u} = \| \nabla f(x_0, y_0) \| \| \vec{u} \| \cos \theta$: (1) The direction of maximum change in a function $f(x, y)$ at a point (x_0, y_0) is the direction of the gradient vector $\nabla f(x_0, y_0)$. The maximum change is $\| \nabla f(x_0, y_0) \|$. (2) The direction of maximum negative change in a function $f(x, y)$ at a point (x_0, y_0) is the direction of $-\nabla f(x_0, y_0)$.

5. Optimisation. $H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

To find and classify critical points. \rightarrow solve $\nabla f = 0 \rightarrow$ critical point(s) (x_0, y_0) \rightarrow substitute (x_0, y_0) into Hessian of f .

If $\det H_f(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then a minimum.

If $\det H_f(x_0, y_0) < 0$ and $f_{xx}(x_0, y_0) < 0$, then a maximum.

If $\det H_f(x_0, y_0) < 0$ then a saddle point.

6. Constrained optimisation - Lagrange Multipliers.

To find the maximum and minimum of the function $f(x, y)$ subject to the constraint, $g(x, y) = 0$:

1) Find the points (x, y) which satisfy the equations $\nabla f = \lambda \nabla g$

$$\begin{cases} f_x(x, y) = \lambda g_x(x, y) \\ f_y(x, y) = \lambda g_y(x, y) \\ g(x, y) = 0 \end{cases}$$

2) Evaluate $f(x, y)$ to find if it's maximum or minimum.

If $f(x, y, z)$ are subject to two constraints. $\nabla f = \lambda \nabla g + \mu \nabla h$, $g(x, y, z) = 0$ and $h(x, y, z) = 0$.

Sequences and series.

1. sequences.

Geometric sequence r^n . when $0 < r < 1$, $\lim_{n \rightarrow \infty} r^n = 0$; when $-1 < r < 0$, $\lim_{n \rightarrow \infty} r^n = 0$; when $r = 1$, $\lim_{n \rightarrow \infty} r^n = 1$; when $r < -1$ or $r > 1$, limit DNE.

Limit taking techniques: suppose two sequences a_n and b_n converge and k is a constant.

$$\lim_{n \rightarrow \infty} k a_n = k \lim_{n \rightarrow \infty} a_n$$

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} (a_n \times b_n) = \lim_{n \rightarrow \infty} a_n \times \lim_{n \rightarrow \infty} b_n$$

Squeezing theorem: if $b_n \leq a_n \leq c_n$ and $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} a_n = L$. (useful in sequences involving $\sin(n)$, $\cos(n)$, $(-1)^n$).

L'Hôpital's rule: used to find the limits of indeterminate forms such as $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0 , ∞^0 , 1^∞ .

(1) If necessary, rearrange expression into the $\frac{f(n)}{g(n)}$ form so that $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}$ becomes $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

(2) Use $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$.

Note: limits of polynomial fractions $\lim_{n \rightarrow \infty} \frac{a_n^{k+1}}{b_n^{l+1}} = \frac{a}{b}$ if $k=l$; 0 if $k < l$; ∞ if $k > l$.

If the sequence has indeterminate form 0^0 , ∞^0 or 1^∞ : find $\lim_{n \rightarrow \infty} \ln a_n = b$, then $\lim_{n \rightarrow \infty} a_n = e^b$.

e.g. $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \ln n^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} = 0$. Hence $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = e^0 = 1$.

Note that in general, $\lim_{n \rightarrow \infty} (1 + \frac{a}{n})^{bn} = e^{ab}$.

2. series

Geometric series: $a \sum_{n=0}^{\infty} r^n$ note: (1) the starting index for geometric series is 0. (2) $r = \frac{a_{n+1}}{a_n} = \text{a constant ratio}$ (3) $a = \text{first term}$.

$$\sum_{n=0}^{\infty} r^n = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} \text{ if } |r| < 1 \text{ or otherwise diverges.}$$

Ratio test for convergence:

for $\sum_{n=0}^{\infty} a_n$, if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, the series converges; > 1 , the series diverges; $= 1$, inconclusive.

Note the exception: hyperharmonic or p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

3. Taylor polynomials: The Taylor polynomial of degree k for $f(x)$ about a point c is the polynomial $p_k(x)$ of degree k such that $p_k(x) = f(c) + f'(c)(x-c) + f''(c) \frac{(x-c)^2}{2!} + \dots + f^{(k)}(c) \frac{(x-c)^k}{k!} = \sum_{n=0}^k \frac{f^{(n)}(c)(x-c)^n}{n!}$

When $c=0$, the Maclaurin polynomial of degree k : $p_k(x) = f(0) + f'(0)x + f''(0) \frac{x^2}{2!} + \dots = \sum_{n=0}^k \frac{f^{(n)}(0)x^n}{n!}$

4. Taylor series: The Taylor series of a function f about the center c is $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!}$

When $c=0$, the Taylor series becomes the Maclaurin series.

Both the Taylor series and the Maclaurin series are power series.

A power series $\sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} a_n x^n$ is convergent when $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ (the ratio test).

Taylor series from known formulae. $\frac{a}{Bx+C} = \frac{a}{1-Bx-C}$ by solving for a and b = $a \sum_{n=0}^{\infty} [b(x-c)]^n$ when $C=0$.

$\frac{A}{Bx+C} = \frac{A}{1-Bx-C}$ by solving for a and b = $a \sum_{n=0}^{\infty} \frac{[b(x-c)]^n}{n!}$ when $C \neq 0$.

e^x centred at $c = e^c \sum_{n=0}^{\infty} \frac{(x-c)^n}{n!}$.