

example: Let f be denoted by $f(x_1, x_2, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2$

The level sets of $f(\vec{x}) = k$, $k > 0$ in \mathbb{R}^n are called $(n-1)$ -spheres, denoted by S^{n-1}

Week 2 Definition of a Limit for functions of one variable

$$\lim_{x \rightarrow a} f(x) = L$$

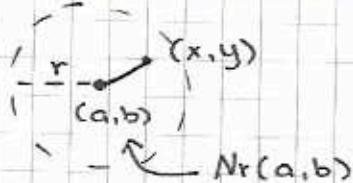
For every $\epsilon > 0$ there exists $\delta > 0$ s.t. $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$ and $\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$

For a scalar function $f(x,y)$, we want $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ to mean that the values of $f(x,y)$ can be made close to L by taking (x,y) sufficiently close to (a,b) .

Open interval $(-r, r) = \{x : |x| < r\}$ for $r \in \mathbb{R}$

An r -neighbourhood of a point $(a,b) \in \mathbb{R}^2$ is a set

$$Nr(a,b) = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y) - (a,b)\| < r, r \in \mathbb{R}\}$$



$\|(x,y) - (a,b)\| = \sqrt{(x-a)^2 + (y-b)^2}$ is Euclidean distance

Limit: Assume $f(x,y)$ is defined in a neighbourhood of (a,b) , except possibly at (a,b) . If for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < \|(x,y) - (a,b)\| < \delta \rightarrow |f(x,y) - L| < \epsilon$$

then $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$

If $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ and $\lim_{(x,y) \rightarrow (a,b)} g(x,y)$ both exist then

$$\hookrightarrow \lim_{(x,y) \rightarrow (a,b)} (f(x,y) + g(x,y)) = \lim_{(x,y) \rightarrow (a,b)} f(x,y) + \lim_{(x,y) \rightarrow (a,b)} g(x,y)$$

$$\hookrightarrow \lim_{(x,y) \rightarrow (a,b)} (f(x,y)g(x,y)) = \lim_{(x,y) \rightarrow (a,b)} f(x,y) \cdot \lim_{(x,y) \rightarrow (a,b)} g(x,y)$$

$$\hookrightarrow \lim_{(x,y) \rightarrow (a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y) \rightarrow (a,b)} f(x,y)}{\lim_{(x,y) \rightarrow (a,b)} g(x,y)} \text{ provided } \lim_{(x,y) \rightarrow (a,b)} g(x,y) \neq 0$$

If $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ exists then the limit is unique.

example: $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$. Prove that \lim doesn't exist.

First, approach a limit along $y=0$

$$\lim_{(x,y) \rightarrow (0,0)} f(x,0) = \lim_{x \rightarrow 0} \frac{0x}{x^2+0} = 0$$

Now approach it along the line $y=x$

$$\lim_{(x,y) \rightarrow (0,0)} f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

Since $f(x,y)$ approaches different values as (x,y) tends to $(0,0)$ along different paths, \lim dne.

example: $f(x,y) = \frac{|x|}{|x|+y^2}$ for $(x,y) \neq 0$. Show that $\lim_{(x,y) \rightarrow (0,0)} f(x,0) = 1$
but $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ dne

Approaching a lim along $y=mx$, we get $\lim_{(x,y) \rightarrow (0,0)} f(x,mx) = \lim_{x \rightarrow 0} \frac{|x|}{|x|+(mx)^2} =$

$$= \lim_{x \rightarrow 0} \frac{|x|}{|x|+m^2x^2} = \frac{x}{x+m^2x^2} = \frac{1}{1+m^2} = 1$$

$$\lim_{x \rightarrow 0^+} \frac{|x|}{|x|+m^2x^2} = \lim_{x \rightarrow 0^+} \frac{x}{x+m^2x^2} = \frac{1}{1+m^2} = 1$$

$$\lim_{x \rightarrow 0^-} \frac{|x|}{|x|+m^2x^2} = \lim_{x \rightarrow 0^-} \frac{-x}{-x+m^2x^2} = \frac{1}{1+m^2} = 1$$

$$\text{Hence, } \lim_{(x,y) \rightarrow (0,0)} f(x,mx) = 1$$

On the other hand, if we approach along $x=0$, we get

$$\lim_{(x,y) \rightarrow (0,0)} \frac{|x|}{|x|+y^2} = \lim_{y \rightarrow 0} \frac{|0|}{|0|+y^2} = 0$$

Hence, \lim dne because $f(x,y)$ doesn't approach to the same value as $(x,y) \rightarrow (0,0)$

example: $f(x,y) = \frac{x^2y}{x^4+y^2}$ for $(x,y) \neq (0,0)$. Show that $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ dne.

Test limit along $x=0$ and $y=mx$

$$\text{For } y=mx, \lim_{x \rightarrow 0} \frac{x^2mx}{x^4+m^2x^2} = \frac{mx}{(x^2+m^2)} = 0$$

$$\text{For } x=0 \quad \lim_{y \rightarrow 0} \frac{0y}{y^2} = 0$$

$$\text{For } y=x^2 \quad \lim_{x \rightarrow 0} \frac{x^2 x^2}{x^4 + x^4} = \lim_{x \rightarrow 0} \frac{x^4}{2x^4} = \frac{1}{2} //$$

So $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist

Squeeze Theorem

If there exists function $B(x,y)$ st $|f(x,y) - L| \leq B(x,y) \quad \forall (x,y) \neq (a,b)$ in some neighbourhood of (a,b) and $\lim_{(x,y) \rightarrow (a,b)} B(x,y) = 0$ then

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$$

example Prove $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} = 0$

$$f(x,y) = \frac{x^2 y}{x^2 + y^2} \quad L = 0 \quad \text{So for } (x,y) \neq (0,0) \text{ we get}$$

$$|f(x,y) - L| = \left| \frac{x^2 y}{x^2 + y^2} - 0 \right| = \frac{x^2 |y|}{x^2 + y^2}$$

Since $y^2 \geq 0$, it follows that $x^2 \leq x^2 + y^2$ so

$$\frac{x^2 |y|}{x^2 + y^2} \leq \frac{(x^2 + y^2) |y|}{x^2 + y^2} = |y| = B(x,y)$$

so we have $0 \leq |f(x,y) - L| \leq |y|$

$\lim_{(x,y) \rightarrow (0,0)} B(x,y) = 0$ so by Squeeze Theorem, \lim exists

example Prove $\frac{|2x^2 - y^2|}{|x| + |y|} \leq 2|x| + |y| \quad \forall (x,y) \neq (0,0)$

$$|2x^2 - y^2| = |2x^2 + (-y^2)| \leq |2x^2| + |-y^2| = 2|x^2| + |y^2|$$

Since $|x| \leq |x| + |y|$ and $|y| \leq |x| + |y|$ we obtain

$$2|x|^2 + |y|^2 \leq 2|x|(|x| + |y|) + |y|(|x| + |y|) = (2|x| + |y|)(|x| + |y|)$$

$$\text{Hence } \frac{|2x^2 - y^2|}{|x| + |y|} \leq \frac{(2|x| + |y|)(|x| + |y|)}{|x| + |y|} = 2|x| + |y| \quad \square$$

example $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - |x| - |y|}{|x| + |y|}$

$$\text{Try } y = mx \quad \lim_{x \rightarrow 0} \frac{x^2 - |x| - |mx|}{|x| + |mx|} = \frac{|x| - (1+|m|)}{1+|m|} = -1$$

Let's use Squeeze Theorem with $L = -1$

$$\left| \frac{x^2 - |x| - |y|}{|x| + |y|} + 1 \right| = \frac{x^2}{|x| + |y|} = \frac{|x||x|}{|x| + |y|} \leq \frac{|x|(|x| + |y|)}{|x| + |y|} = |x|$$

$$B(x,y) = |x| \quad \lim_{(x,y) \rightarrow (0,0)} B(x,y) = 0$$

So by squeeze theorem $\lim = -1$

If $a < b$ and $c < 0$ then $bc < ac$ Multiplication Property

$|a| < b$ iff $-b < a < b$

$|a+b| \leq |a| + |b|$ Triangle Inequality

$2|x||y| \leq x^2 + y^2$ Cosine Inequality

$$|x| \leq \sqrt{x^2 + y^2}$$

$$\left| \frac{x^3 - y^3}{x^2 + y^2} \right| \leq |x| + |y|$$

A function $f(x,y)$ is continuous at (a,b) iff

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$$

example. $f(x,y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$ if $f(x,y)$ continuous at $(0,0)$

$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} = 0$ since $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0 = f(0,0)$, f is continuous

example. $f(x,y) = \frac{\sin(xy)}{x^2 + y^2}$ if $(x,y) \neq (0,0)$

$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2}$ dne so f isn't continuous

For scalar functions $g(t)$ and $f(x,y)$ the composite function $g \circ f$ is defined by $(g \circ f)(x,y) = g(f(x,y))$ $\forall (x,y) \in D(f)$ for which $f(x,y) \in D(g)$

Continuity Theorem 1: If f and g are both continuous at (a,b) then $f+g$ and fg are continuous at (a,b)

Continuity Theorem 2: If f and g are continuous at (a,b) and $g(a,b) \neq 0$ then the quotient f/g is continuous at (a,b)

Continuity Theorem 3: If $f(x,y)$ is continuous at (a,b) and $g(t)$ is continuous at $f(a,b)$ then $g \circ f$ is continuous at (a,b)

week 3

Partial Derivatives

A scalar function $f(x,y)$ can be differentiated in two natural ways

1) By treating y as a constant and differentiating with respect to x to obtain $\frac{df}{dx}$

2) By treating x as a constant and differentiating with respect to y to obtain $\frac{df}{dy}$

The derivatives $\frac{df}{dx}$ and $\frac{df}{dy}$ are called the first partial derivatives

The partial derivatives of $f(x,y)$ are defined by

$$\frac{df}{dx}(x,y) = f_x(x,y) = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{df}{dy}(x,y) = f_y(x,y) = \lim_{h \rightarrow 0} \frac{f(x,y+h) - f(x,y)}{h}$$

provided that these limits exist.

example. $f(x,y) = xe^{kxy}$.

$$\frac{df}{dx} = e^{kxy} + xe^{kxy}ky = (1+kxy)e^{kxy}$$

$$\text{and } \frac{df}{dy} = xe^{kxy}kx = kx^2e^{kxy}$$

example Determine whether $\frac{df}{dx}(0,0)$ exists for $f(x,y) = (x^3+y^3)^{1/3}$

$$\frac{df}{dx} = \frac{1}{3}(x^3+y^3)^{-2/3} 3x^2 = \frac{x^2}{(x^3+y^3)^{2/3}}$$

$\frac{df}{dx}$ DNE at $\frac{df}{dx}(0,0)$

To find $\frac{df}{dx}(0,0)$ we must use the definition of partial derivatives

$$\frac{df}{dx}(x,y) = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h} = \frac{(h^3+0^3)^{1/3} - 0}{h} = 1$$

Therefore, $\frac{df}{dx}(0,0)$ exists and $\frac{df}{dx}(0,0) = 1$

$$\text{example. } f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h} = \frac{\frac{h(0)}{h^2} - 0}{h} = 0$$

$$\lim_{h \rightarrow 0} \frac{f(0, 0+h) - f(0, 0)}{h} = \frac{\frac{0h}{h^2} - 0}{h} = 0$$

The concept of partial derivatives do not match our concept of differentiability for functions of one variable, where differentiability implies continuity.

example. $f(x, y) = |x(y-1)|$

$$f_x(0, 0) \rightarrow \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \frac{|h(-1)| - 0}{h} = \frac{|h|}{h} = \pm 1 \text{ DNE}$$

$$f_x(0, 1) \rightarrow \lim_{h \rightarrow 0} \frac{f(0+h, 1) - f(0, 1)}{h} = \lim_{h \rightarrow 0} \frac{|h(0)| - 0}{h} = 0$$

example. $f(x, y, z) = xy^2z^3$

$$f_x = y^2z^3$$

$$f_y = 2xz^3y$$

$$f_z = 3xy^2z^2$$

Second Partial Derivatives

$$\frac{d^2 f}{dx^2} = d_{xx} = D_1^2 f \quad \frac{d^2 f}{dy dx} = d_{xy} = D_2 D_1 f$$

$$\frac{d^2 f}{dx dy} = f_{yx} = D_1 D_2 f \quad \frac{d^2 f}{dy^2} = d_{yy} = D_2^2 f$$

example. $f(x, y) = xe^{kxy}$

$$\frac{df}{dx} = d_x = e^{kxy} + xe^{kxy}ky$$

$$\frac{df}{dy} = d_y = xe^{kxy}kx = x^2ke^{kxy}$$

$$\frac{d^2 f}{dx^2} = d_{xx} = e^{kxy}ky + e^{kxy}ky + xe^{kxy}k^2y^2$$

$$\frac{d^2 f}{dy dx} = d_{xy} = \frac{d}{dy}(e^{kxy} + xe^{kxy}ky) = (e^{kxy}kx + xke^{kxy} + xkye^{kxy}kx)$$

$$= 2kxe^{kxy} + x^2k^2ye^{kxy}$$

$$\frac{d^2 f}{dx dy} = d_{yx} = \frac{d}{dx}(x^2ke^{kxy}) = 2xke^{kxy} + x^2ke^{kxy}ky$$

$$\frac{d^2 f}{dy^2} = d_{yy} = \frac{d}{dy}(x^2ke^{kxy}) = x^2ke^{kxy}kx$$

Clairaut's Theorem

If f_{xy} and f_{yx} are defined in some neighborhood of (a,b) and are both continuous at (a,b) then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

The tangent plane to $z = f(x,y)$ at the point $(a,b, f(a,b))$

is
$$z = f(a,b) + \frac{df}{dx}(a,b)(x-a) + \frac{df}{dy}(a,b)(y-b)$$

example. $f(x,y) = \sqrt{x^2+y^2}$ at $P(3,-4,5)$

$$\frac{df}{dx}(x,y) = f_x = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}$$

$$\frac{df}{dy}(x,y) = f_y = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}$$

$$z = 5 + \frac{3}{5}(x-3) - \frac{4}{5}(y+4)$$

The tangent line for $y=f(x)$ at the point $(a,f(a))$ is
$$y = f(a) + f'(a)(x-a)$$

For a function $f(x,y)$ we define the linearization $L(a,b)(x,y)$ of f at (a,b) by
$$L(a,b)(x,y) = f(a,b) + \frac{df}{dx}(a,b)(x-a) + \frac{df}{dy}(a,b)(y-b)$$

We call the approximation $f(x,y) \approx L(a,b)(x,y)$ the linear approximation of $f(x,y)$ at (a,b)

example. approximate $\sqrt{0.95^3 + 1.98^3}$

$$f(x,y) = \sqrt{x^3+y^3} \text{ and } (a,b) = (1,2)$$

$$f(a,b) = \sqrt{1^3+2^3} = 3$$

$$\frac{df}{dx}(a,b) = \frac{1}{2}(x^3+y^3)^{-\frac{1}{2}} \cdot 3x^2 = \frac{3 \cdot 1}{2 \cdot 3} = \frac{1}{2}$$

$$\frac{df}{dy}(a,b) = \frac{1}{2}(x^3+y^3)^{-\frac{1}{2}} \cdot 3y^2 = \frac{3 \cdot 4}{2 \cdot 3} = 2$$

$$L(a,b) = 3 + \frac{1}{2}(x-1) + 2(y-2) \approx 3 + \frac{1}{2}(0.95-1) + 2(1.98-2) = 2.935$$

The actual value is 2.935943

example. approximate $\sqrt{\sin(\frac{1}{10}) + \tan(\frac{3}{4})}$

$$f(x,y) = \sqrt{\sin x + \tan y} \quad (a,b) = (0, \pi/4)$$

$$f(0, \pi/4) = \sqrt{0+1} = 1$$

$$f_x = \frac{1}{2} (\sin x + \tan y) (\cos x + \frac{\cos^2 y}{\sin^2 y}) = \frac{1}{2}(1)(1 + \frac{0}{1}) = \frac{1}{2}$$

$$f_y(0, \pi/4) = 1$$

$$L(0, \pi/4) = 1 + \frac{1}{2}x + (y - \pi/4) = 1 + \frac{1}{20} + \frac{3}{4} - \frac{\pi}{4} = 1.015$$

example. An isosceles triangle has a base $x=4$ m and angles $\pi/4$. The base was increased by 16 cm and angles were decreased by 0.1, estimate the change in the area.

Let x be the length of a base of an isosceles triangle, θ is the measure of angles, and h is the height.

$$\text{The area is } f(x, \theta) = \frac{1}{2}xh = \left(\frac{1}{2}x \tan \theta\right) \frac{1}{2}x = \frac{1}{4}x^2 \tan \theta$$

$$\text{Recall that } \Delta f = \frac{df}{dx}(a,b)(\Delta x) + \frac{df}{dy}(a,b)(\Delta y)$$

The change in x is $\Delta x = 16 \text{ cm} = 0.16 \text{ m}$ and $\Delta \theta = -0.1$

$$f_x = \frac{1}{2}x \tan \theta \quad f_\theta = \frac{1}{4}x^2 \sec^2 \theta$$

$$f_x(4, \pi/4) = 2 \quad f_\theta(4, \pi/4) = 8$$

Using the increment form of linear approximation, we have

$$\Delta f = 2 \cdot 0.16 + 8 \cdot (-0.1) = -0.48$$

∴ The area decreased by -0.48

Suppose $f(x,y,z)$ has partial derivatives at $\vec{a} \in \mathbb{R}^3$. The gradient of f at \vec{a} is defined by $\nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a}))$

Suppose that $f(\vec{x})$, $\vec{x} \in \mathbb{R}^3$, has partial derivatives at $\vec{a} \in \mathbb{R}^3$

The linearization of f at \vec{a} is defined by

$$L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a})(\vec{x} - \vec{a})$$

The linear approximation of f at \vec{a} is

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a})(\vec{x} - \vec{a})$$

example. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ find the gradient

and the linear approximation for f at $\vec{a} = (1, 2, -2)$

$$\nabla f(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right)$$

Now, evaluate $\nabla f(x,y,z)$ at $\vec{a} = (1, 2, -2)$

$$\nabla f(\vec{a}) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right)$$

$$\text{Thus, } L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 3 + \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right) \cdot (x-1, y-2, z+2)$$

$$(x-1, y-2, z+2) = 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) - \frac{2}{3}(z+2)$$

$$\Delta f \sim \nabla f(\vec{a}) \cdot \vec{\Delta x}$$

$$\Delta f \sim \nabla f(a,b) \cdot \Delta(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

example. $f(x,y) = \begin{cases} \frac{\sin(xy)}{\ln(x^2+y^2+1)} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

$$f_x(0,0) = \frac{\ln(x^2+y^2+1) \cos(xy)y - \frac{1 \cdot 2x \sin(xy)}{x^2+y^2+1}}{\ln(x^2+y^2+1)^2} \quad \text{dne}$$

$$\lim_{h \rightarrow 0} \frac{\sin((x+h)y)}{\ln((x+h)^2+y^2+1)} - 0 = 0$$

$$f_y(0,0) = \frac{\sin(0(h+0))}{\ln(h^2+1)} = 0$$

example. The temperature of a metal rod at position x , $0 \leq x \leq 1$, and at time t , $t \geq 0$ is given by $u(t,x) = 100e^{-t} \sin(\pi x)$. Find the rate of change with respect to position when $x = 3/4$, $t = 1$

$$u_x(t,x) = 100\pi e^{-t} \cos(\pi x)$$

$$u_x(1, 3/4) = -81.72$$

$$u_t(t,x) = -100e^{-t} \sin(\pi x)$$

$$u_t(1, 3/4) = -26.01$$

example. find all planes that are tangent to the surface $f(x,y) = 1 - x^2 - y^2$ at $(a,b, f(a,b))$ and contain the line passing through the points $(1, 0, 2)$ and $(0, 2, 2)$. The equation of a tangent line is $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$